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FINITE ELEMENT METHOD

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Module 11: Locking

Lecture 52 : Shear Locking

Concepts Covered

- **Locking in 1D beam**
- **Parasitic Shear in Q4 – Shear locking**

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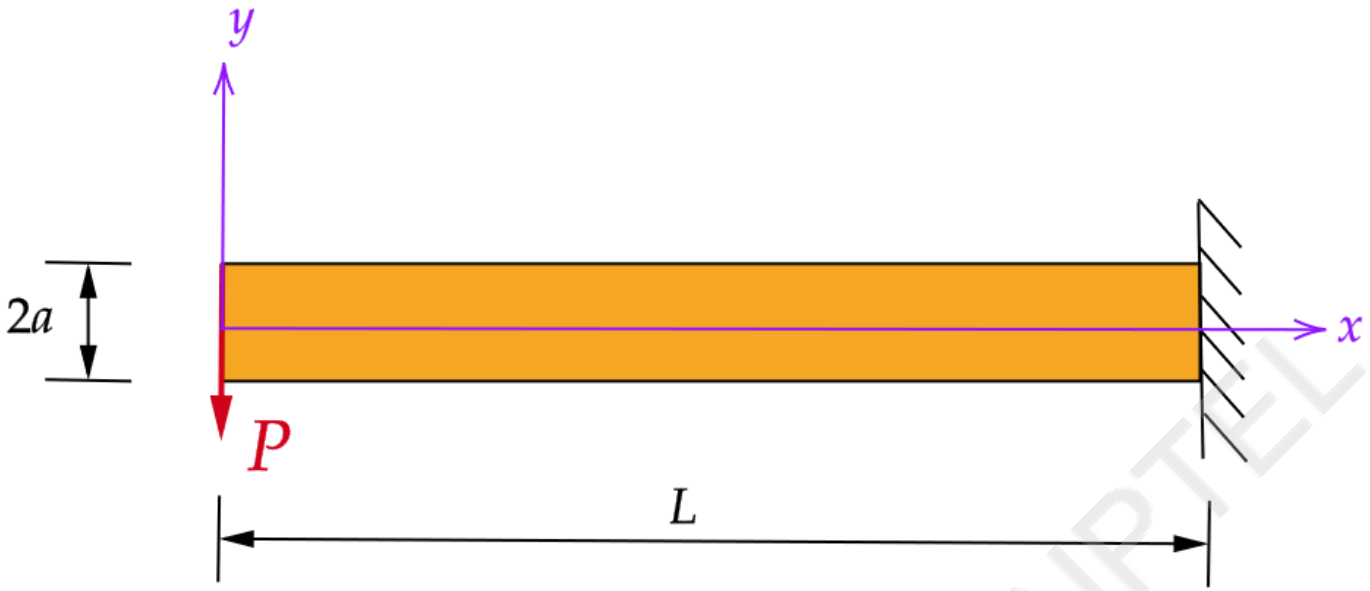
Cantilever Beam Problem



$$u(x,y) = \frac{Px^2y}{2EI} + \nu\frac{Py^3}{6EI} - \frac{Py^3}{6IG} - \left(\frac{PL^2}{2EI} - \frac{Pa^2}{2IG}\right)y$$

$$v(x,y) = -\nu\frac{Px^2y}{2EI} - \frac{Px^3}{6EI} + \frac{PL^3x}{2EI} - \frac{PL^3}{3EI}$$

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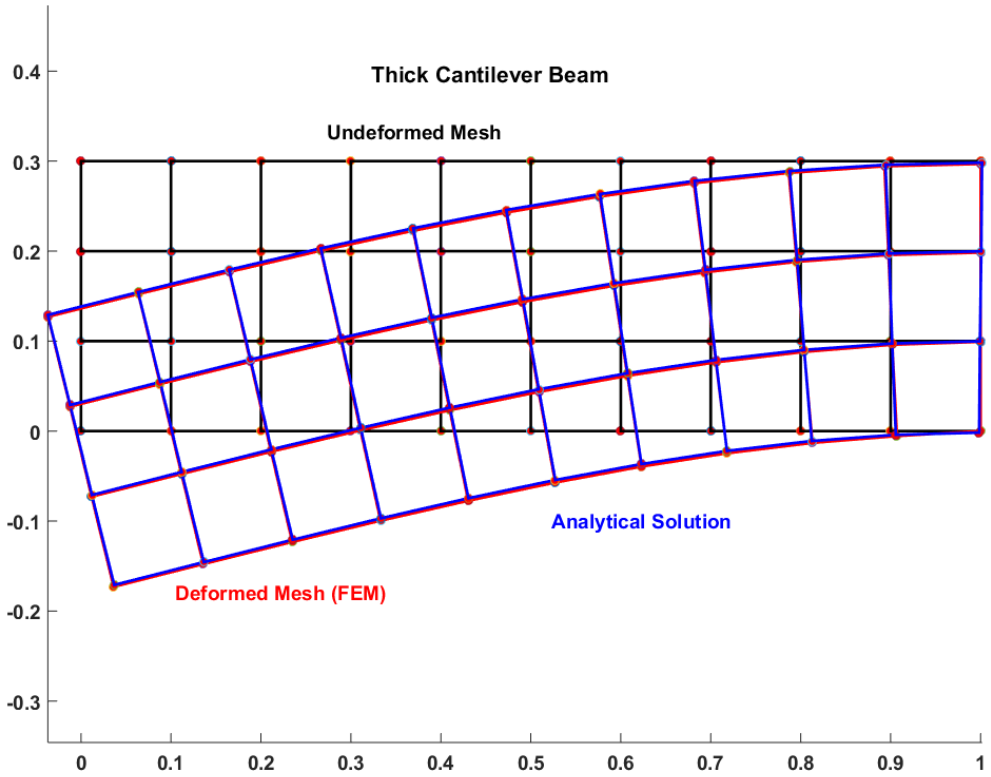
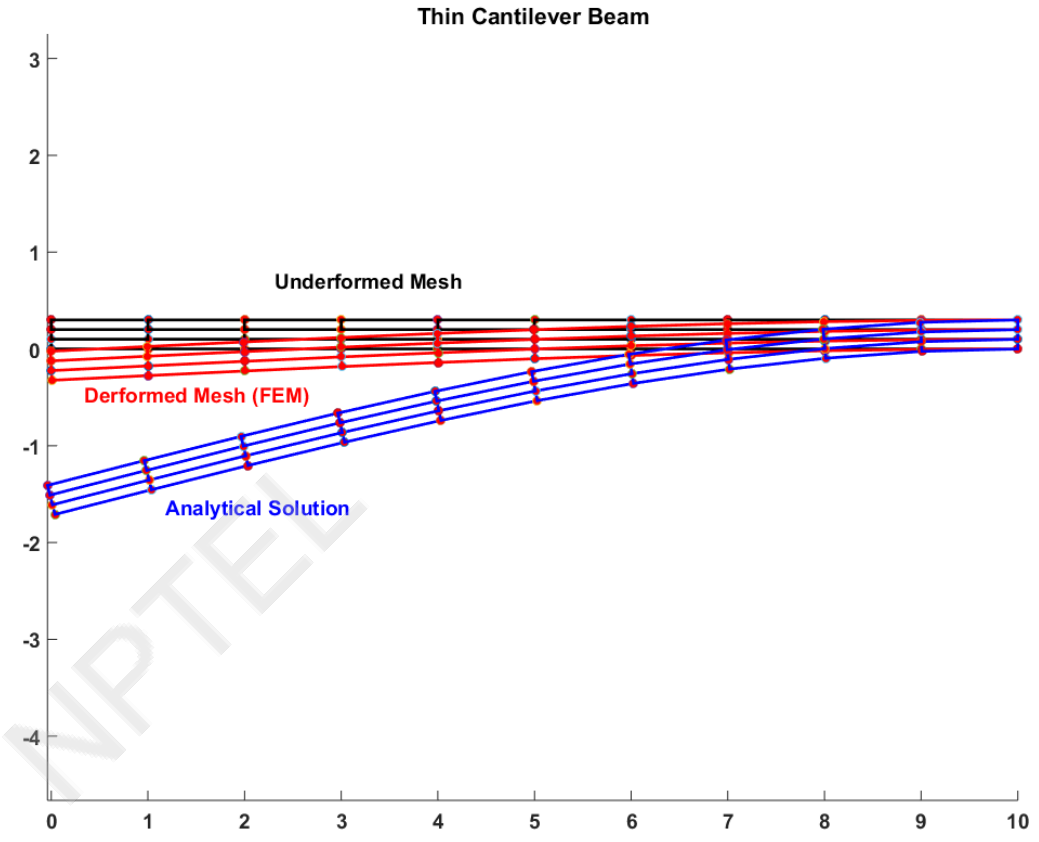


a/L = 0.15



a/L = 0.015

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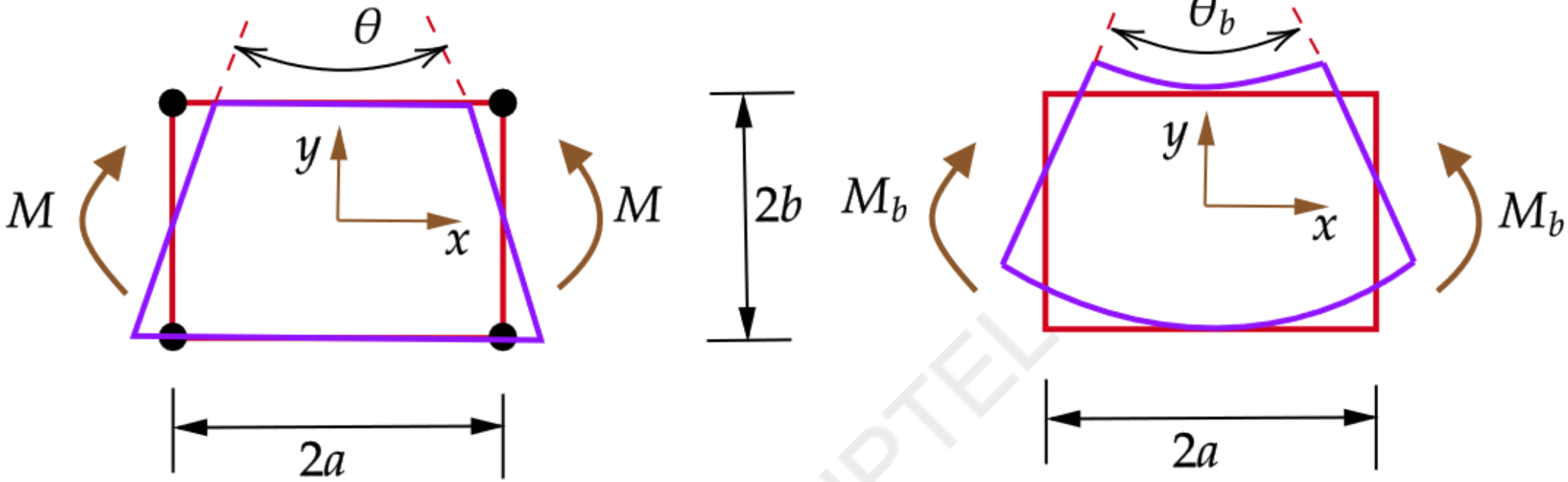
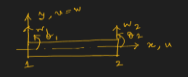

θ
y
x
M
M
2b
2a
θb
y
x
Mb
Mb
2a



$$w(x) = \left(1-\frac{x}{L}\right)w_1 + \frac{x}{L}w_2$$
$$\theta(x) = \left(1-\frac{x}{L}\right)\theta_1 + \frac{x}{L}\theta_2$$

EB → thin beam $\quad \theta \simeq \frac{dw}{dx}$

$$\frac{dw}{dx} = \frac{w_2 - w_1}{L} = \text{const.}$$

$$u(x,y,z) = -y\theta(x)$$
$$v(x,y,z) = w(x)$$

$$\epsilon_{xx} = \frac{\partial u}{\partial x} = -y\frac{d\theta}{dx}$$
$$\epsilon_{yy} = 0$$
$$\gamma_{xy} = \frac{dw}{dx} - \theta(x)$$

$$\sigma_{xx} = -E y \frac{d\theta}{dx}$$
$$\sigma_{yy} = 0$$
$$\sigma_{xy} = K_s G\left(\frac{dw}{dx} - \theta\right)$$

$A = bh$
$I = \frac{bh^3}{12}$

$$U = U_b + U_s$$
$$= \frac{1}{2}\int_0^L EI\left(\frac{d\theta}{dx}\right)^2 dx + \frac{K_s G A}{2}\int_0^L \left(\frac{dw}{dx} - \theta\right)^2 dx$$
$$= \frac{Ebh^3}{24}\int_0^L \left(\frac{d\theta}{dx}\right)^2 + \frac{K_s G bh}{2}\int_0^L \left(\frac{dw}{dx} - \theta\right)^2 dx$$

$$w(x) = \frac{x}{L}w_2$$
$$\theta(x) = \frac{x}{L}\theta_2$$

$w_1 = 0$
$\theta_1 = 0$

$$U_b = \frac{Ebh^3}{24}\int_0^L \frac{\theta_2^2}{L^2}dx = \frac{Ebh^3\theta_2^2 L}{24L^2} = \frac{Ebh^3\theta_2^2}{24L}$$

$$U_s = \frac{K_s G bh}{2}\int_0^L \left(\frac{w_2}{L} - \frac{x\theta_2}{L}\right)^2 dx = \frac{K_s G bh}{2L^2}\int_0^L \left(w_2^2 + x^2\theta_2^2 - 2xw_2\theta_2\right)dx$$
$$= \frac{K_s G bh}{2L^2}\left(w_2^2 L + \frac{L^3\theta_2^2}{3} - L^2 w_2\theta_2\right)$$

$$\pi = \frac{Ebh^3\theta_2^2}{24L} + \frac{K_s G bh}{2L^2}\left(w_2^2 L + \frac{L^3\theta_2^2}{3} - L^2 w_2\theta_2\right) - Pw_2$$

$$\frac{\partial \pi}{\partial w_2} = 0, \quad \frac{\partial \pi}{\partial \theta_2} = 0$$

$$\frac{\partial \pi}{\partial w_2} = 0 = \frac{K_s G bh}{2L^2}\left(2w_2 L - L^2\theta_2\right) - P = 0 \quad \text{— (a)}$$

$$\frac{\partial \pi}{\partial w_2} = 0 = \frac{Ebh^3\theta_2}{24L} + \frac{K_s G bh}{2L^2}\left(\frac{2}{3}\theta_2 L^3 - L^2 w_2\right) = 0 \quad \text{— (b)}$$

From (b) →

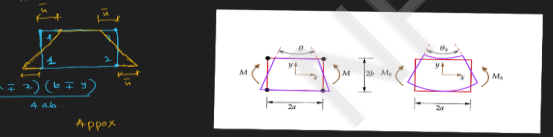
$$\theta_2 = \frac{K_s G bh}{2L}w_2\left[\frac{1}{\frac{Ebh^3}{12L^2} + \frac{K_s G bh}{3}}\right] = \frac{K_s G A}{2L}w_2\left[\frac{1}{\frac{EA}{12}\left(\frac{h}{L}\right)^2 + \frac{K_s G A}{3}}\right]$$

Sub. θ_2 in (a) →

$$w_2 = \left(\frac{E\left(h/L\right)^2 + 4K_s G A}{E\left(h/L\right)^2 + K_s G A}\right)\frac{PL}{K_s G A}$$

Thin beam $\frac{h}{L} \ll 1$ $\quad \frac{h}{L} \simeq 0$ $\quad w_2 = 4\left(\frac{PL}{K_s G A}\right) = 4{\cdot}8\frac{PL}{GA}$ $\quad K_s = \frac{5}{6}$

Thick beam $\frac{h}{L} \simeq 1$ $\quad w_2 = \left(\frac{E + 4K_s G A}{E + K_s G A}\right)\frac{PL}{K_s G A}$



$$u = a_0 + a_1 x + a_2 y + a_3 xy$$
$$v = b_0 + b_1 x + b_2 y + b_3 xy$$

$$N_i = \frac{(a \mp x)(b \mp y)}{4ab}$$

Appox.

$$\bar{u} = \frac{\theta}{2}b$$

Exact $\quad \bar{u} = \frac{\theta_b}{2}b$

$$u(x,y) = -\frac{xy}{ab}\bar{u} = -\frac{xy}{2a}\theta$$
$$v(x,y) = 0$$

$$u(x,y) = -\frac{xy}{2a}\theta_b = 0$$
$$v(x,y) = -\left(1-\frac{x^2}{a^2}\right)\frac{a}{2b}\cdot\frac{\theta_b}{2}\cdot b$$
$$= \theta\left(1-\frac{y^2}{b^2}\right)\frac{b}{2a}\cdot\frac{\theta_b}{2}\cdot b$$

$$\epsilon_{xx} = -\frac{\theta y}{2a}$$
$$\epsilon_{yy} = 0$$
$$\gamma_{xy} = -\frac{\theta x}{2a}$$

$$\begin{Bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{xy}\end{Bmatrix} = \frac{E}{1-\nu^2}\begin{bmatrix}1 & \nu & 0\\ \nu & 1 & 0\\ 0 & 0 & \frac{1-\nu}{2}\end{bmatrix}\begin{Bmatrix}\epsilon_{xx}\\ \epsilon_{yy}\\ \gamma_{xy}\end{Bmatrix}$$

$\{\sigma\}$ $\quad \{\epsilon\}$

$$\epsilon_{xx} = -\frac{\theta_b y}{2a}$$
$$\epsilon_{yy} = \nu\epsilon_{xx}$$
$$\gamma_{xy} = 0$$

$$\sigma_{yy} = 0$$
$$\sigma_{xy} = 0$$

$$U_a = \frac{1}{2}\int_V \{\sigma\}^T\{\epsilon\}\,dv$$
$$= \frac{1}{2}\int\int\int \frac{E}{1-\nu^2}\epsilon_{xx}^2\,dx\,dy\,dz + \frac{1}{2}\int\int\int \frac{E}{1-\nu^2}\cdot\frac{1-\nu}{2}\gamma_{xy}^2\,dx\,dy\,dz$$
$$= \frac{1}{2}\frac{Eh}{1-\nu^2}\int_{-b}^{b}\int_{-a}^{a}\frac{\theta^2 y^2}{4a^2}\,dx\,dy + \frac{E}{1-\nu^2}\cdot\frac{h}{2}\cdot\frac{1-\nu}{2}\int_{-b}^{b}\int_{-a}^{a}\frac{\theta^2 x^2}{4a^2}\,dx\,dy$$
$$= \frac{Ehb^3\theta^2}{6a(1-\nu^2)} + \frac{E}{1-\nu^2}\left(\frac{abh(1-\nu)}{12}\theta^2\right)$$

$$U_{exact} = \frac{1}{2}\int \{\sigma\}^T\{\epsilon\}\,dv$$
$$= \frac{1}{2}\int_{-h/2}^{h/2}\int_{-b}^{b}\int_{-a}^{a} E\epsilon_{xx}^2\,dx\,dy\,dz$$
$$= \frac{h}{2}\int_{-b}^{b}\int_{-a}^{a} E\left(\frac{\theta_b^2 y^2}{4a^2}\right)dx\,dy$$
$$= \frac{Ebh^3\theta_b^2}{6a}$$

$$\frac{U_{approx/ele}}{U_{exact}} = \frac{\frac{Ehb^3\theta^2}{6a(1-\nu^2)} + \frac{E}{1-\nu^2}\left(\frac{abh(1-\nu)}{12}\theta^2\right)}{\frac{Ebh^3\theta_b^2}{6a}}$$
$$= \frac{\theta^2}{\theta_b^2}\cdot\frac{1}{1+\nu}\left[\frac{1}{1-\nu} + \frac{1}{2}\left(\frac{\epsilon}{b}\right)^2\right]$$

$$\frac{M}{M_b} = \frac{1}{1+\nu}\left[\frac{1}{1-\nu} + \frac{1}{2}\left(\frac{a}{b}\right)^2\right]$$

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Module 11: Shear Locking

Lecture 53 : Selective reduced Integration and Modes of Q4 Element

Concepts Covered

- Parasitic Shear in Q4 – Shear locking
- Selective Reduced Integration
- Modes of Q4 Element

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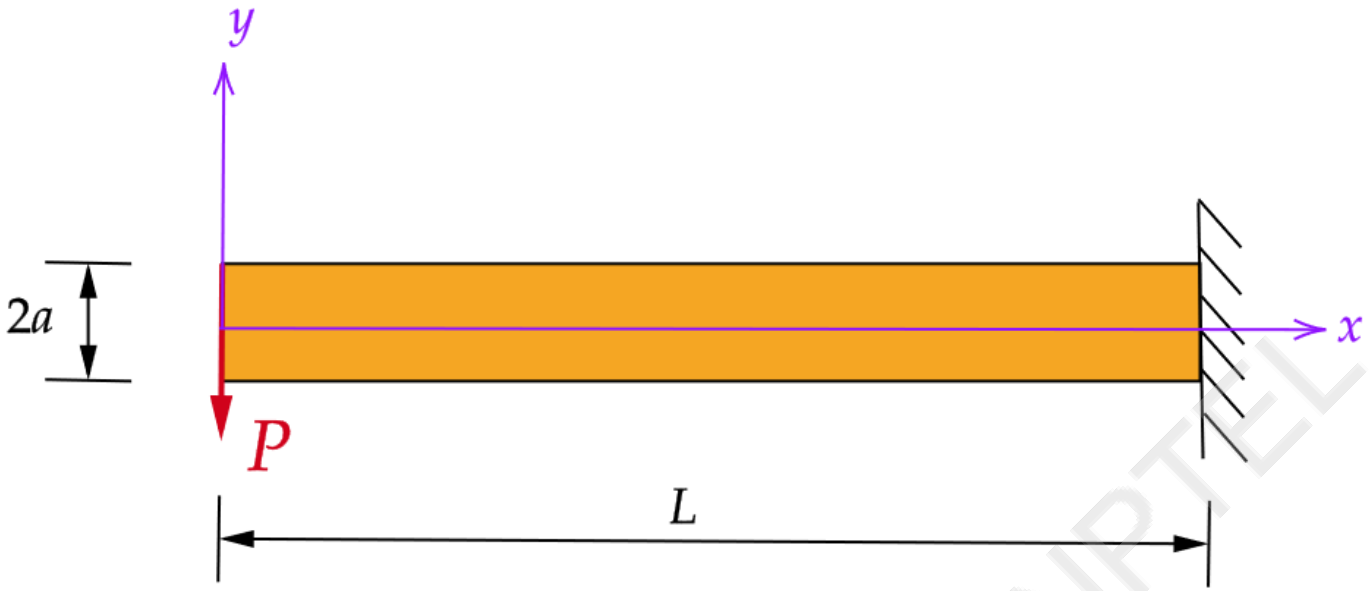
Cantilever Beam Problem



$$u(x,y) = \frac{Px^2y}{2EI} + \nu\frac{Py^3}{6EI} - \frac{Py^3}{6IG} - \left(\frac{PL^2}{2EI} - \frac{Pa^2}{2IG}\right)y$$

$$v(x,y) = -\nu\frac{Px^2y}{2EI} - \frac{Px^3}{6EI} + \frac{PL^3x}{2EI} - \frac{PL^3}{3EI}$$

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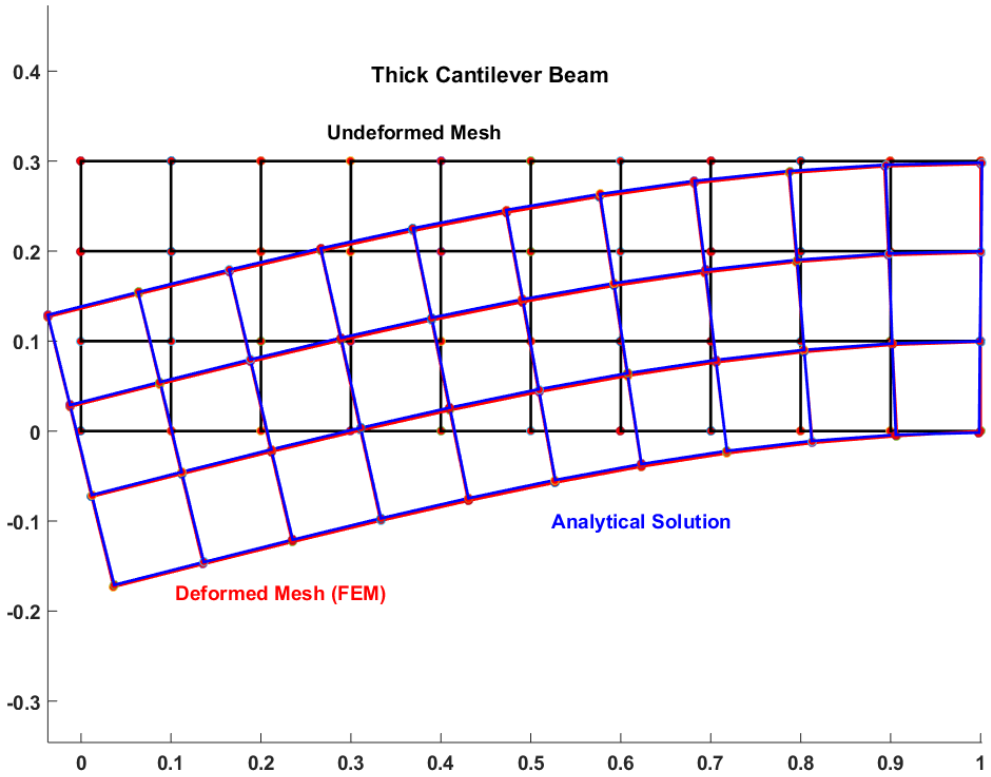
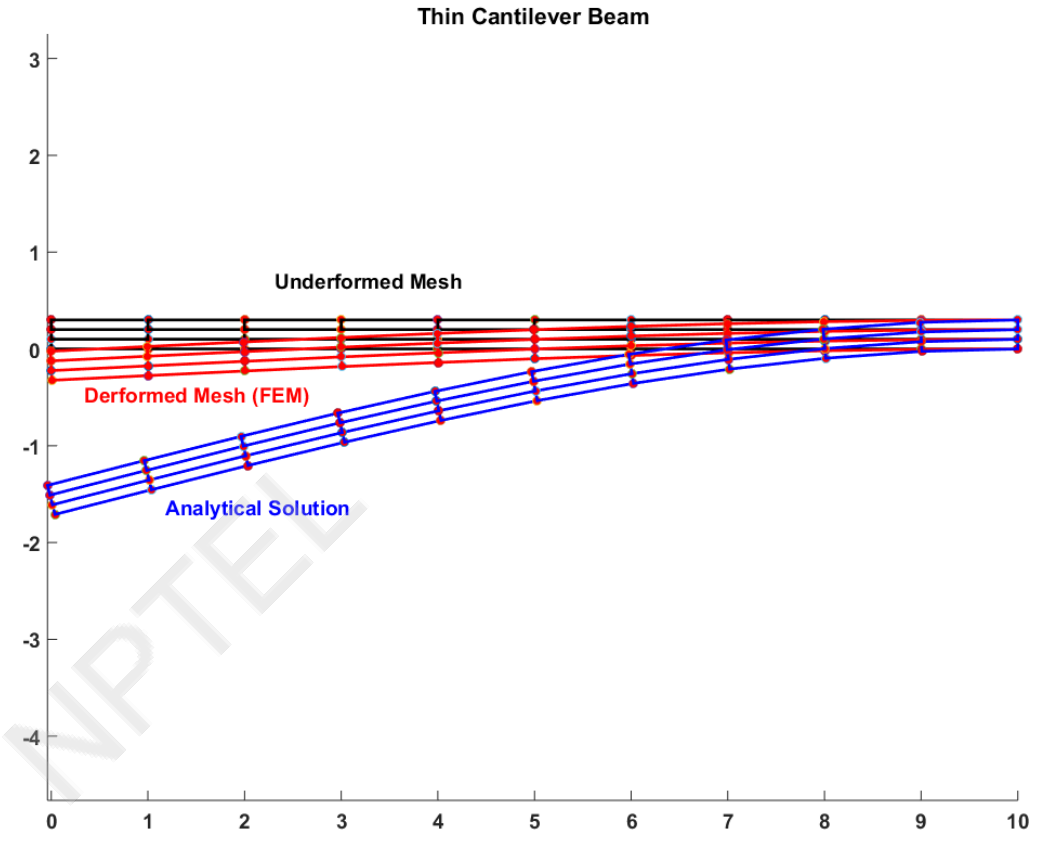

Thick Cantilever Beam
Undeformed Mesh
Analytical Solution
Deformed Mesh (FEM)
0.4
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1


Thin Cantilever Beam
Underformed Mesh
Derformed Mesh (FEM)
Analytical Solution
3
2
1
0
-1
-2
-3
-4
0
1
2
3
4
5
6
7
8
9
10

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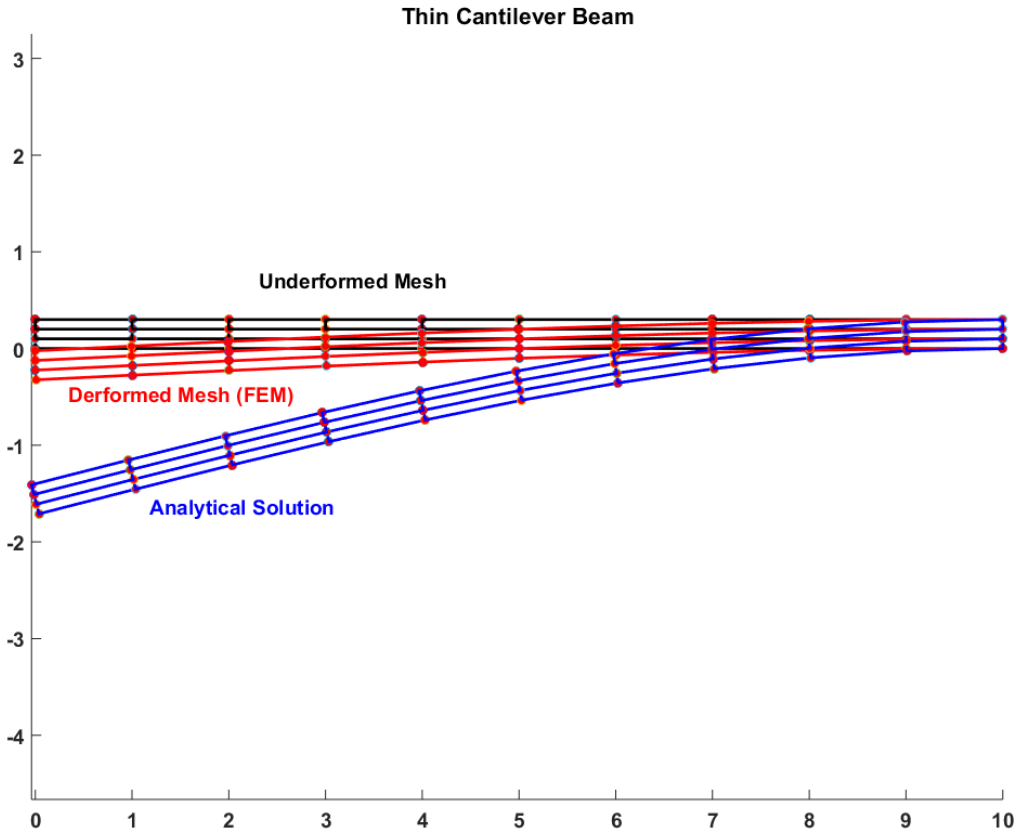
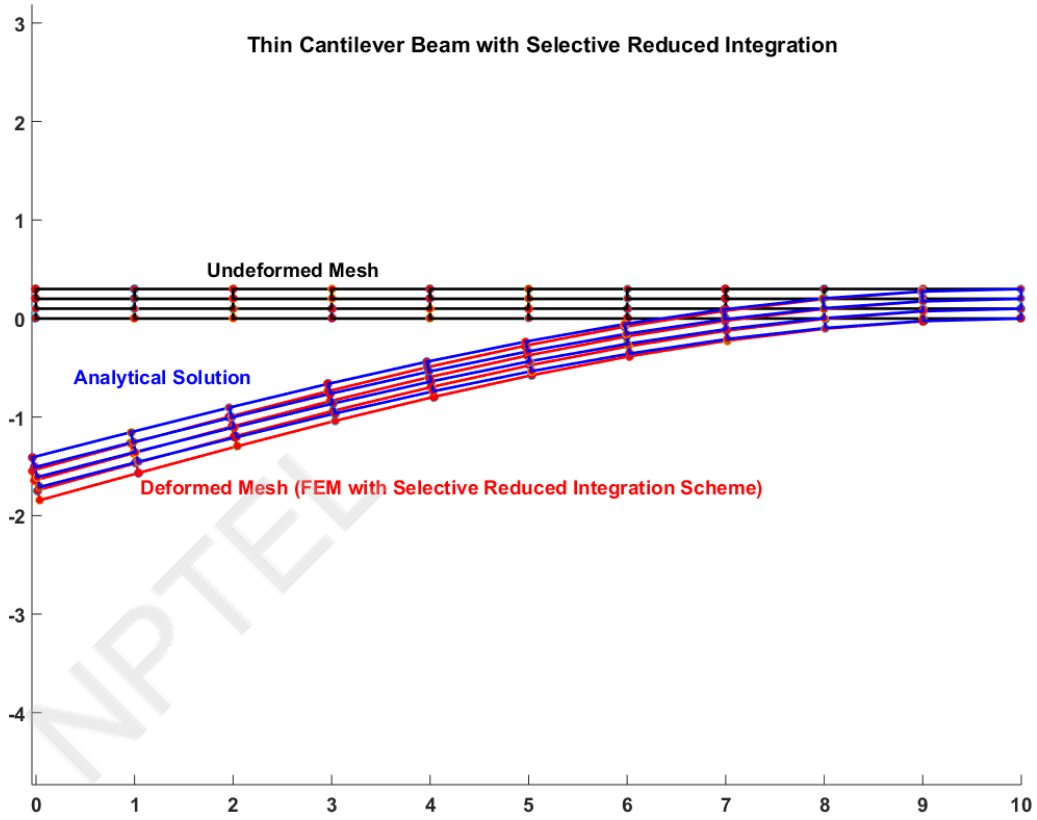

Thin Cantilever Beam
Underformed Mesh
Derformed Mesh (FEM)
Analytical Solution


Thin Cantilever Beam with Selective Reduced Integration
Undeformed Mesh
Analytical Solution
Deformed Mesh (FEM with Selective Reduced Integration Scheme)

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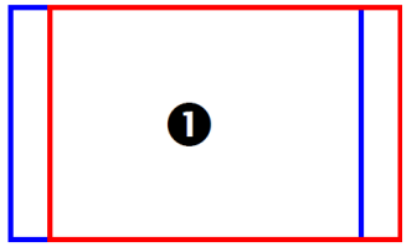
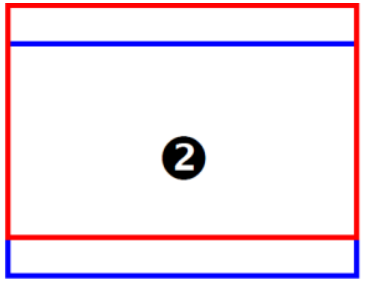

1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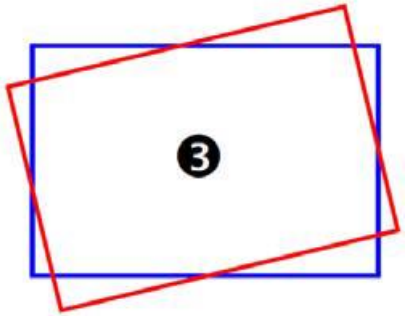
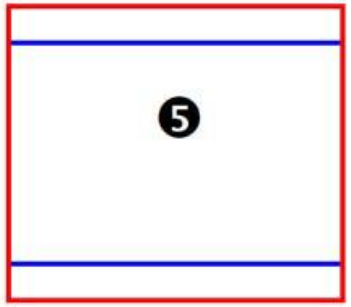

3

4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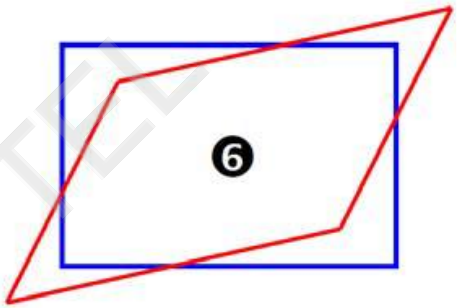
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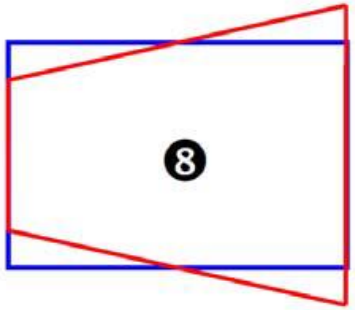
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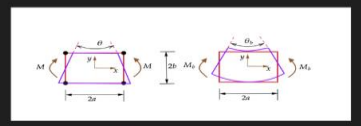


$$\frac{U_{approx.}}{U_{exact}} = \frac{B^2}{D_b^2} \cdot \frac{1}{1+\nu}\left[\frac{1}{1-\nu} + \frac{1}{2}\left(\frac{a}{b}\right)^2\right]$$

$$\frac{M}{M_b} = \frac{1}{1+\nu}\left[\frac{1}{1-\nu} + \frac{1}{2}\left(\frac{a}{b}\right)^2\right]$$

as $\left(\frac{a}{b}\right) \to \infty$ $\quad \frac{M}{M_b} \to \infty$

$$\epsilon_{xx} = -\frac{\beta y}{2a}, \quad \epsilon_{yy} = 0, \quad \gamma_{xy} = -\frac{\beta x}{2a}$$

at $z = 0$, $\gamma_{xy} = 0$ ← Expected

$$K^e = \int_V B^T C B \, dv$$

$$C = \begin{bmatrix} c_{11} & c_{12} & 0 \\ c_{12} & c_{22} & 0 \\ 0 & 0 & c_{66} \end{bmatrix}$$

$$= \int_V B^T C_b B \, dv + \int_V B^T C_s B \, dv$$

$$= \begin{bmatrix} c_{11} & c_{12} & 0 \\ c_{12} & c_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & c_{66} \end{bmatrix}$$

$$= K_b + K_s \qquad = C_b + C_s$$

2×2 $(\xi, \eta = \pm\frac{1}{\sqrt{3}})$

1×1 $(\xi = 0, \eta = 0)$

K_b → full integration 2×2

K_s → reduced integration 1×1

Modes of Q4 Element :–

$$K^e = \int_V B^T C B \, dv$$

full integration.

$\text{rank}(K^e) = 5$

$= 8 - 3$ ← no. of rigid body modes.

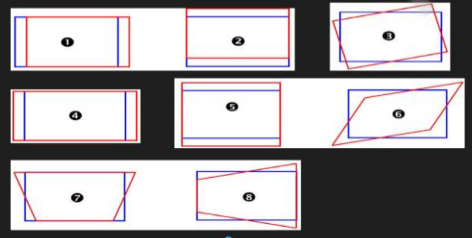
reduced integration rank $= 3$

$= 8 - 3 - 2$ ← Hourglass modes.

↑ rigid body modes

$\to \det(K - \lambda I) = 0 \to$

$$U = \frac{1}{2}\int \{\epsilon\}^T [C] \{\epsilon\} \, dv$$



① → $u = c$, $v = 0$ → $\epsilon_{xx} = 0$, $\epsilon_{yy} = 0$, $\gamma_{xy} = 0$

② → $u = 0$, $v = c$ → $\epsilon_{xx} = \epsilon_{yy} = \gamma_{xy} = 0$

③ → $u = c\eta$, $v = -c\xi$ $\epsilon_{xx} = 0$, $\epsilon_{yy} = 0$

$\gamma_{xy} = c - c = 0$

④ → $u = c\xi$, $v = 0$ → $\epsilon_{xx} = c$, $\epsilon_{yy} = 0$, $\gamma_{xy} = 0$

⑤ → $u = 0$, $v = c\eta$ → $\epsilon_{xx} = 0$, $\epsilon_{yy} = c$, $\gamma_{xy} = 0$

⑥ → $u = c\eta$, $v = c\xi$ → $\epsilon_{xx} = 0$, $\epsilon_{yy} = 0$

$\gamma_{xy} = 2c$

⑦ → $u = c\xi\eta$, $v = 0$, $\epsilon_{xx} = c\eta$, $\epsilon_{yy} = 0$

$\gamma_{xy} = c\xi$

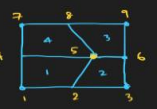
⑧ → $u = 0$ $v = c\xi\eta$. $\epsilon_{xx} = 0$, $\epsilon_{yy} = c\xi$

$\gamma_{xy} = c\eta$

1×1

2×2

① x y



u	v	
1	0	①
0	1	②
x	0	④
0	x	⑦
y	0	⑤
0	y	⑥

Eg. Engg. version Patch Test

① displacements at ε should be exact.

② gradient of displacement should represent constant strain

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Module 11: Shear Locking

Lecture 54 : Incompatible Elements

Concepts Covered

- What is Incompatible Elements ?
- Q6 and QM6 Element

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References:

1. Wilson, E. L., R. L. Taylor, W. P. Doherty, and J. Ghaboussi. "Incompatible displacement models." In *Numerical and computer methods in structural mechanics*, pp. 43-57. Academic Press, 1973.
2. Taylor, Robert L., Peter J. Beresford, and Edward L. Wilson. "A non-conforming element for stress analysis." *International Journal for numerical methods in Engineering* 10, no. 6 (1976): 1211-1219.
3. Taylor, R. L., J. C. Simo, O. C. Zienkiewicz, and ACH Chan. "The patch test—a condition for assessing FEM convergence." *International journal for numerical methods in engineering* 22, no. 1 (1986): 39-62.
4. Ibrahimbegovic, Adnan, and E. L. Wilson. "A modified method of incompatible modes." *Communications in Applied Numerical Methods* 7, no. 3 (1991): 187-194.

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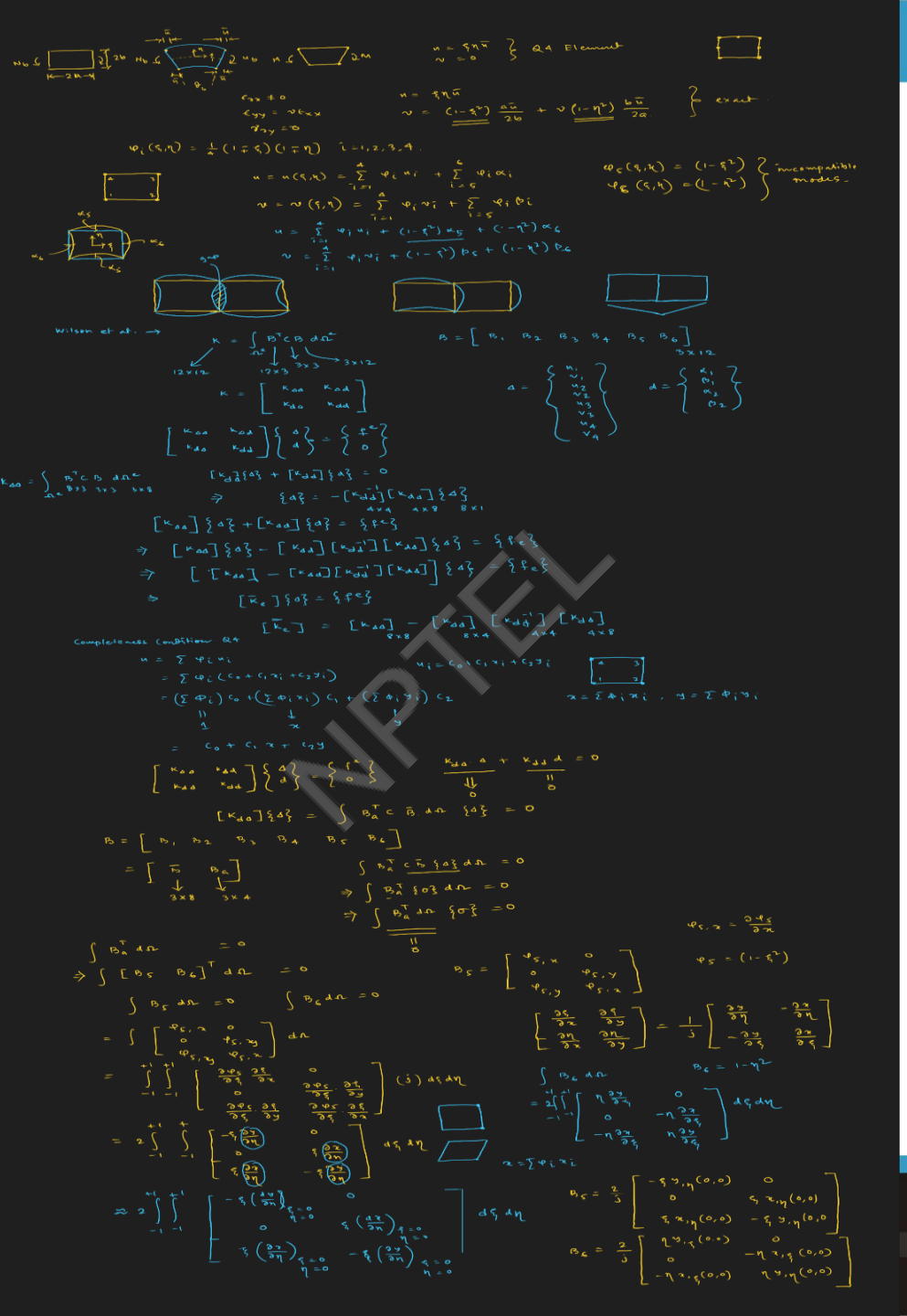
Q4 Element
exact
incompatible modes
Wilson et al. →
Completeness condition Q4
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FINITE ELEMENT METHOD

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Module 11: Volumetric Locking

Lecture 55 : Nearly Incompressible Material

Concepts Covered

- **Volumetric Locking**
- **Selective Reduced Integration**

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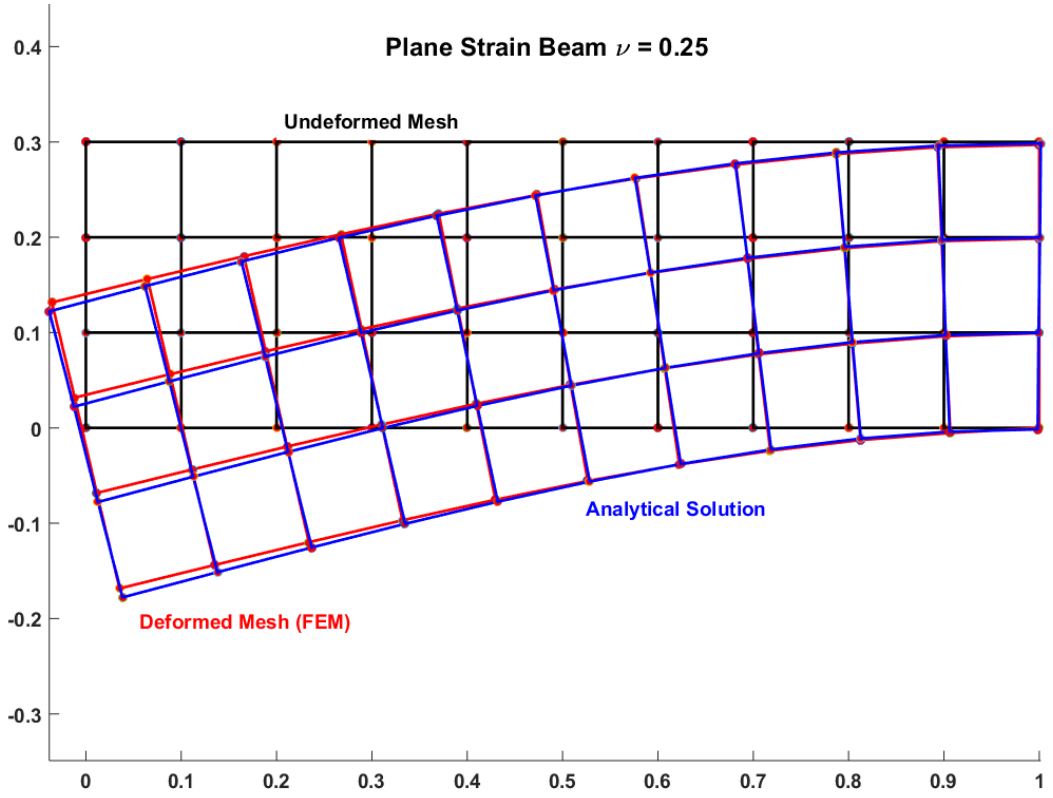
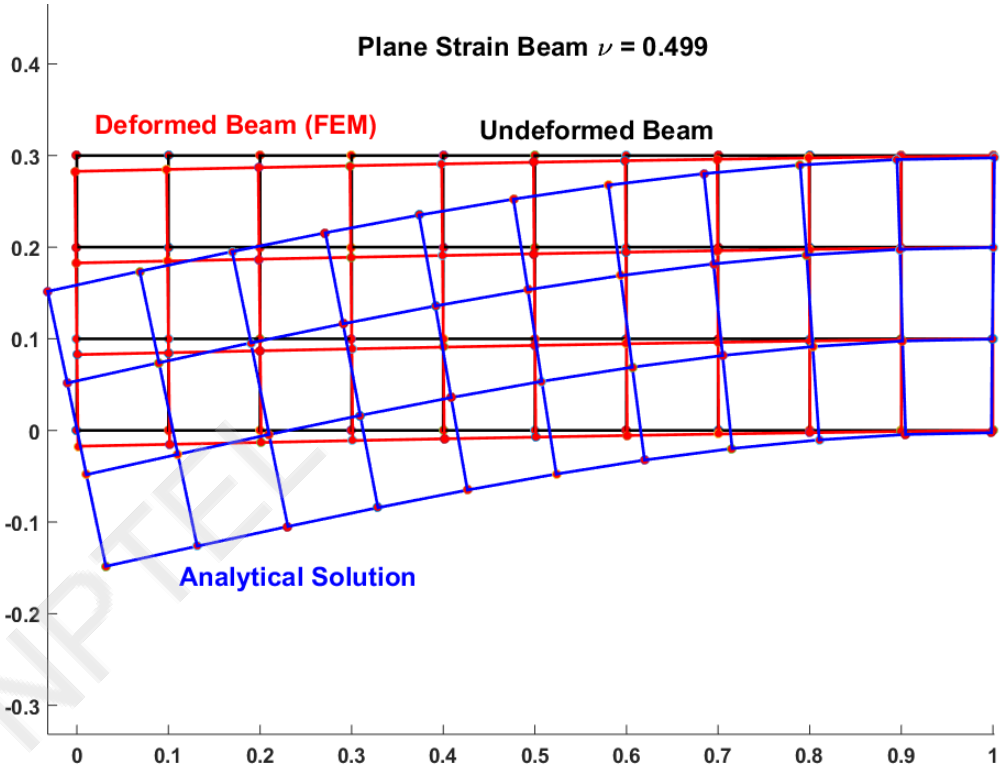

Plane Strain Beam ν = 0.25
Undeformed Mesh
Analytical Solution
Deformed Mesh (FEM)
0.4
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1


Plane Strain Beam ν = 0.499
Deformed Beam (FEM)
Undeformed Beam
Analytical Solution
0.4
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

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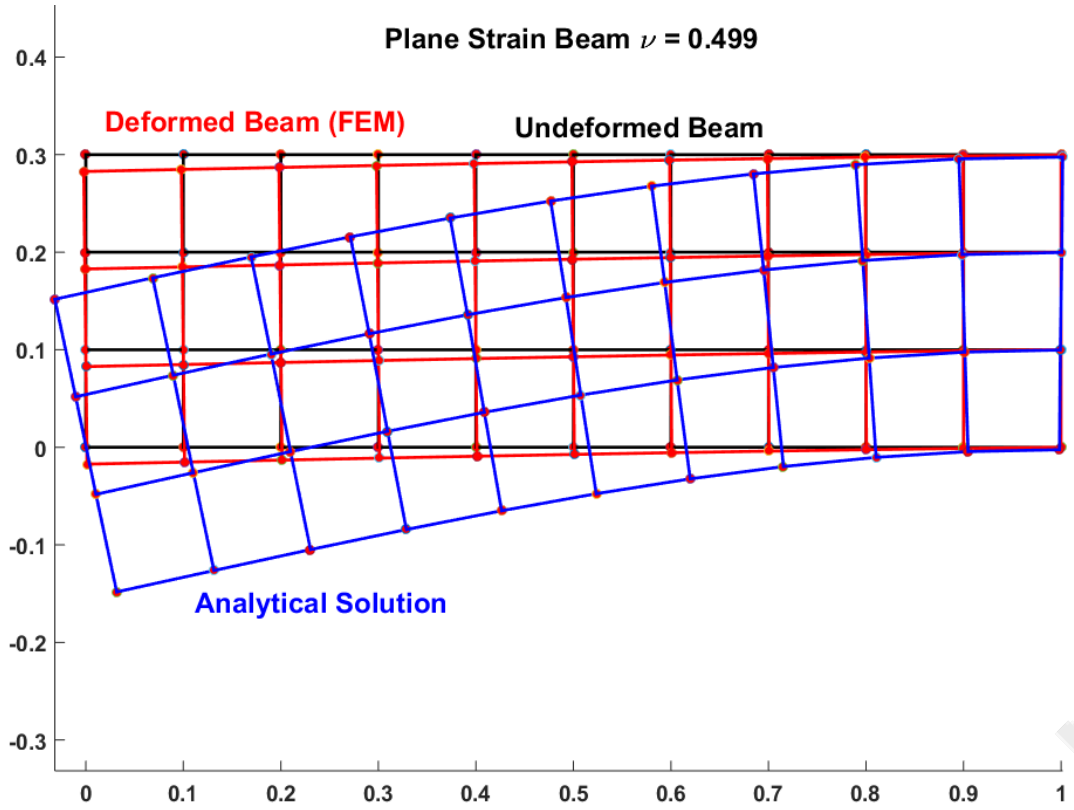
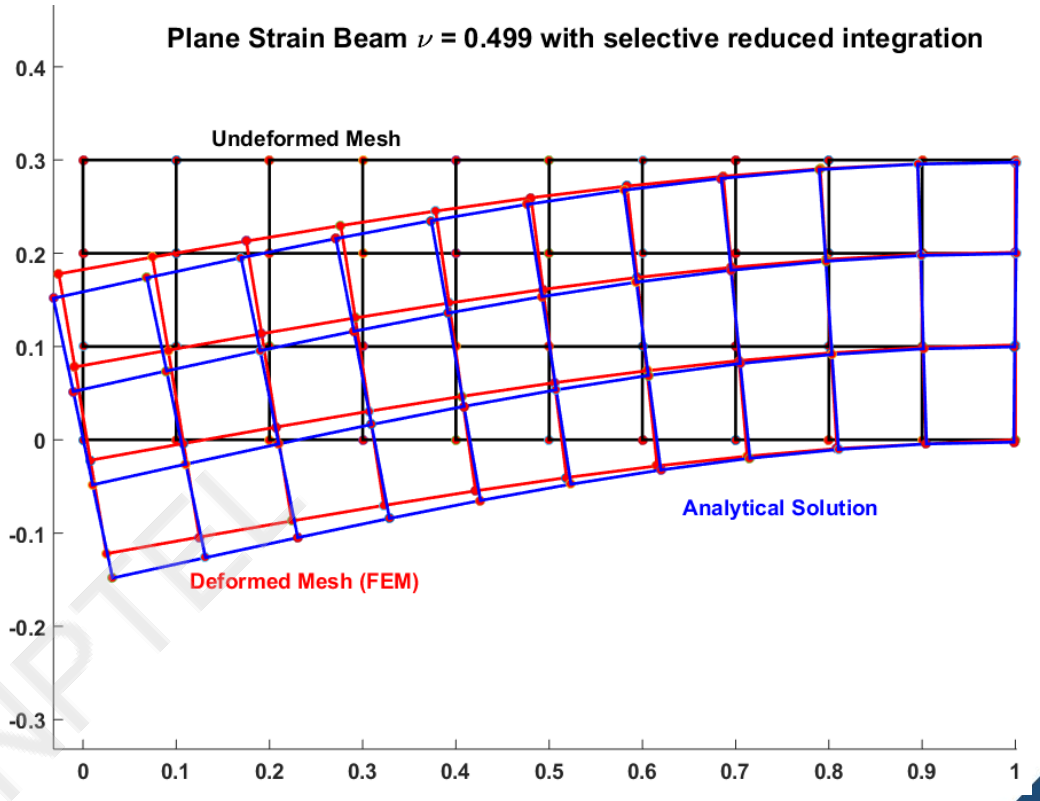

Plane Strain Beam ν = 0.499
Deformed Beam (FEM)
Undeformed Beam
Analytical Solution
0.4
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1


Plane Strain Beam ν = 0.499 with selective reduced integration
Undeformed Mesh
Analytical Solution
Deformed Mesh (FEM)
0.4
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

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Isotropic Case:

plane strain case:

$$C = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix}$$

$\nu \to 0.5$

$$C = \begin{bmatrix} \lambda+2\mu & \lambda & 0 \\ \lambda & \lambda+2\mu & 0 \\ 0 & 0 & \mu \end{bmatrix}$$

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)} \qquad \mu = \frac{E}{2(1+\nu)}$$

$\frac{\lambda}{\mu} \to \infty$ as $\nu \to 0.5$

$$C = \begin{bmatrix} K+\frac{4}{3}\mu & K-\frac{2}{3}\mu & 0 \\ K-\frac{2}{3}\mu & K+\frac{4}{3}\mu & 0 \\ 0 & 0 & \mu \end{bmatrix}$$

K = Bulk modulus $= \frac{E}{3(1-2\nu)}$

$$= K \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \mu \begin{bmatrix} \frac{4}{3} & -\frac{2}{3} & 0 \\ -\frac{2}{3} & \frac{4}{3} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= K C_K + \mu C_\mu$$

$$C = \begin{bmatrix} \lambda+2\mu & \lambda & 0 \\ \lambda & \lambda+2\mu & 0 \\ 0 & 0 & \mu \end{bmatrix} = \lambda \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \mu \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \lambda C_\lambda + \mu C_\mu$$

$$[K^e] = \int_{\Omega^e} B^T C B \, d\Omega^e$$

$$= \int_{\Omega^e} B^T \lambda C_\lambda B \, d\Omega^e + \int B^T \mu C_\mu B \, d\Omega^e$$

$$= \lambda \int_{\Omega^e} B^T C_\lambda B \, d\Omega^e + \mu \int B^T C_\mu B \, d\Omega^e$$

$$= \lambda [K^e_\lambda] + \mu [K^e_\mu]$$

$$[K^e]\{u^e\} = \{f^e\}$$

$$\left(\lambda [K^e_\lambda] + \mu [K^e_\mu]\right)\{u^e\} = \{s^e\}$$

$$\left(\lambda [K_\lambda] + \mu [K_\mu]\right)\{u\} = \{F\}$$

$$\Rightarrow \quad \{u\} = \left(\lambda [K_\lambda] + \mu [K_\mu]\right)^{-1} \{F\}$$

$$= \left[\lambda \left\{ [K_\lambda] + \frac{\mu}{\lambda} [K_\mu] \right\}\right]^{-1} \{F\}$$

$$= \frac{1}{\lambda} \left[[K_\lambda] + \frac{\mu}{\lambda} [K_\mu] \right]^{-1} \{F\}$$

$$\approx \frac{1}{\lambda} \left[[K_\lambda] \right]^{-1} \{F\}$$

$\nu \to 0.5$, $\frac{\lambda}{\mu} \to \infty$

$$\frac{\lambda}{\mu} = \frac{E\nu}{(1+\nu)(1-2\nu)} \cdot \frac{2(1+\nu)}{E} = \frac{2\nu}{1-2\nu}$$

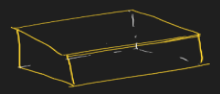
$\nu = 0.5 - 10^{-6}$

$$K = \left(\lambda [K_\lambda] + \mu [K_\mu]\right)$$

$\frac{\lambda}{\mu} >>> 1$

11 → 1×1 ; $[K_\mu]$ → 2×2

2×2
1×1



$$C = \begin{bmatrix} \lambda+2\mu & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda+2\mu & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & \lambda+2\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu \end{bmatrix}_{6\times6} = \lambda C_\lambda + \mu C_\mu$$

$$K = \lambda K_\lambda + \mu K_\mu$$

$1\times1\times1$; $2\times2\times2$

$$K = B K_B + \mu K_\mu$$

1×1 ; 2×2

$$B = \frac{E}{3(1-2\nu)} \qquad \nu \to 0.5, \quad B \to \infty$$

nonlinear: $J = \det(F) \approx 1$ (F)

$$C = \begin{bmatrix} C_{11} & C_{12} & 0 \\ C_{12} & C_{22} & 0 \\ & & \end{bmatrix}$$

$E_1, E_2, \nu_{12}, G_{12}$

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THANK YOU !

NPTEL ONLINE CERTIFICATION COURSES

FINITE ELEMENT METHOD

Biswanath Banerjee & Amit Shaw
Department of Civil Engineering, IIT Kharagpur

Module 11: Volumetric Locking

Lecture 56 : B-Bar Method

Concepts Covered

- **B-Bar method – Strain projection method**

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$$\{\varepsilon\} = \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix} = \begin{bmatrix} \partial/\partial x & 0 & 0 \\ 0 & \partial/\partial y & 0 \\ 0 & 0 & \partial/\partial z \\ \partial/\partial y & \partial/\partial x & 0 \\ 0 & \partial/\partial z & \partial/\partial y \\ \partial/\partial z & 0 & \partial/\partial x \end{bmatrix} \begin{Bmatrix} u \\ v \\ w \end{Bmatrix}$$

$$\varepsilon_v = \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}$$

$$p = \frac{1}{3}(\sigma_{xx} + \sigma_{yy} + \sigma_{zz})$$

$$\{\sigma\} = [N]\{u^e\}$$

$$= \frac{1}{3}\begin{bmatrix} \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{Bmatrix} u \\ v \\ w \end{Bmatrix} + \begin{bmatrix} \frac{2}{3}\partial/\partial x & -\frac{1}{3}\partial/\partial y & -\frac{1}{3}\partial/\partial z \\ -\frac{1}{3}\partial/\partial x & \frac{2}{3}\partial/\partial y & -\frac{1}{3}\partial/\partial z \\ -\frac{1}{3}\partial/\partial x & -\frac{1}{3}\partial/\partial y & \frac{2}{3}\partial/\partial z \\ 0 & \partial/\partial z & \partial/\partial y \\ \partial/\partial z & 0 & \partial/\partial x \\ \partial/\partial y & \partial/\partial x & 0 \end{bmatrix}\begin{Bmatrix} v \\ w \\ u \end{Bmatrix}$$

$$= [B^{dil}]\{u^e\} + [B^{dev}]\{u^e\}$$

$$B^a = \begin{bmatrix} B_1^a & 0 & 0 \\ 0 & B_2^a & 0 \\ 0 & 0 & B_3^a \\ B_2^a & B_1^a & 0 \\ 0 & B_3^a & B_2^a \\ B_3^a & 0 & B_1^a \end{bmatrix} = [B^{dil}] + [B^{dev}]$$

$$\psi_i = \frac{1}{6}(a_i + b_i x + c_i y)(6 \neq 0)$$

$$B_i = \begin{bmatrix} B_i^a \end{bmatrix} \quad a = 1 \text{ to } m \text{ nodes}$$

$$B^e = \begin{bmatrix} B^1 & B^2 & B^3 & \cdots & B^n \\ 6\times3 & 6\times3 & 6\times3 & & 6\times3 \end{bmatrix}_{6\times 3n}$$

$$[N] = \begin{bmatrix} \psi_1 & 0 & 0 & \psi_2 & 0 & 0 & \cdots & \psi_n & 0 & 0 \\ 0 & \psi_1 & 0 & 0 & \psi_2 & 0 & \cdots & 0 & \psi_n & 0 \\ 0 & 0 & \psi_1 & 0 & 0 & \psi_2 & \cdots & 0 & 0 & \psi_n \end{bmatrix}_{3\times 3n}$$

$$B_i^e = \begin{bmatrix} \frac{\partial \psi_i}{\partial x} & 0 & 0 \\ 0 & \frac{\partial \psi_i}{\partial y} & 0 \\ 0 & 0 & \frac{\partial \psi_i}{\partial z} \\ \frac{\partial \psi_i}{\partial y} & \frac{\partial \psi_i}{\partial x} & 0 \\ 0 & \frac{\partial \psi_i}{\partial z} & \frac{\partial \psi_i}{\partial y} \\ \frac{\partial \psi_i}{\partial z} & 0 & \frac{\partial \psi_i}{\partial x} \end{bmatrix}_{6\times 3}$$

$$B_i^e = \begin{bmatrix} b_i & 0 & 0 \\ 0 & c_i & 0 \\ 0 & 0 & d_i \\ c_i & b_i & 0 \\ 0 & d_i & c_i \\ d_i & 0 & b_i \end{bmatrix} = \frac{1}{6}\begin{bmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \\ d_1 & d_2 & d_3 \\ c_1 & c_2 & c_3 \\ d_1 & d_2 & d_3 \\ b_1 & b_2 & b_3 \end{bmatrix} + \begin{bmatrix} \frac{2}{3}b_1 & -\frac{1}{3}b_2 & -\frac{1}{3}b_3 \\ -\frac{1}{3}b_1 & \frac{2}{3}b_2 & -\frac{1}{3}b_3 \\ -\frac{1}{3}b_1 & -\frac{1}{3}b_2 & \frac{2}{3}b_3 \\ 0 & b_3 & b_2 \\ b_3 & 0 & b_1 \\ b_2 & b_1 & 0 \end{bmatrix}$$

$$= [B^{dil}] + [B^{dev}]$$

Bi - bar :-

$$B_i^a = B_i^{dil} + B_i^{dev}$$

$$= \frac{1}{6}\begin{bmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \\ d_1 & d_2 & d_3 \\ c_1 & c_2 & c_3 \\ d_1 & d_2 & d_3 \\ b_1 & b_2 & b_3 \end{bmatrix} + \begin{bmatrix} \frac{2}{3}b_1 & -\frac{1}{3}b_2 & -\frac{1}{3}b_3 \\ -\frac{1}{3}b_1 & \frac{2}{3}b_2 & -\frac{1}{3}b_3 \\ -\frac{1}{3}b_1 & -\frac{1}{3}b_2 & \frac{2}{3}b_3 \\ 0 & b_3 & b_2 \\ b_3 & 0 & b_1 \\ b_2 & b_1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} B_5 & B_2 & B_8 \\ B_4 & B_3 & B_9 \\ B_3 & B_6 & B_{10} \\ B_3 & B_6 & B_{10} \\ B_7 & B_{11} & B_{12} \\ B_2 & B_1 & 0 \end{bmatrix}$$

$$\bar{B}_j = \frac{\int_{\Omega^e} B_j \, d\Omega^e}{\int_{\Omega^e} d\Omega} \quad (j = 1,2,3)$$

$$\bar{B}_1 = \frac{\int_{\Omega^e} B_1 \, d\Omega^e}{\int_{\Omega^e} d\Omega^e}$$

$$B_4 = \frac{1}{3}b_1 - \frac{1}{3}b_2 = \frac{b_1 - b_2}{3}$$

$$B_5 = \frac{2}{3}b_1 - \frac{1}{3}b_2 - \frac{1}{3}b_3 = \frac{1}{3}b_1 - \frac{1}{3}b_2 + \frac{1}{3}b_1 - \frac{1}{3}b_3 = B_4 + B_6$$

$$B_6 = \frac{1}{3}(b_1 - b_3)$$

$$B_7 = B_3 + B_6$$

$$B_8 = \frac{B_3 - B_5}{3}$$

$$B_9 = B_3 + B_8$$

$$\int_{\Omega^e} B^T C \, B \, d\Omega^e \rightarrow \int_{\Omega^e} \bar{B}^T C \, \bar{B} \, d\Omega^e$$

standard — $\bar{B}$ - bar

$$\bar{B}_j = \frac{\int_{\Omega^e} B_j \, d\Omega^e}{\int_{\Omega^e} d\Omega^e}$$

$$= \sum_{i=1}^{n^e} \bar{N}_i \, n_j(\xi_i)$$

plane strain :—

$$\sigma'_{zz} \neq 0 \quad \varepsilon_{zz} = 0$$

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \\ 0 \end{Bmatrix} = \begin{bmatrix} \partial/\partial x & 0 \\ 0 & \partial/\partial y \\ \partial/\partial y & \partial/\partial x \\ 0 & 0 \end{bmatrix}\begin{Bmatrix} u \\ v \end{Bmatrix}_{4\times 2}$$

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{xy} \end{Bmatrix} = \begin{bmatrix} \kappa + \frac{4}{3}\mu & \kappa - \frac{2}{3}\mu & 0 & \kappa - \frac{2}{3}\mu \\ \kappa - \frac{2}{3}\mu & \kappa + \frac{4}{3}\mu & 0 & \kappa - \frac{2}{3}\mu \\ \kappa - \frac{2}{3}\mu & \kappa - \frac{2}{3}\mu & 0 & \kappa - \frac{2}{3}\mu \\ 0 & 0 & \mu & 0 \\ \kappa - \frac{2}{3}\mu & \kappa - \frac{2}{3}\mu & 0 & \kappa + \frac{4}{3}\mu \end{bmatrix}\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \\ 0 \end{Bmatrix}_{4\times 4}$$

standard B_i matrix

$$B_i^a = \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \\ B_2 & B_1 \end{bmatrix}_{3\times 2}$$

$$B^a = \begin{bmatrix} B_1^a & 0 \\ 0 & B_2^a \\ B_2^a & 0 \\ 0 & B_3^a \end{bmatrix}_{4\times 2} \quad a = 1,(1)\,q$$

$$= \frac{1}{3}\begin{bmatrix} B_1^a & B_2^a \\ B_1^a & B_2^a \\ 0 & 0 \\ B_1^a & B_2^a \end{bmatrix} + \begin{bmatrix} \frac{2}{3}B_1^a & -\frac{1}{3}B_2^a \\ -\frac{1}{3}B_1^a & \frac{2}{3}B_2^a \\ -\frac{1}{3}B_1^a & -\frac{1}{3}B_2^a \\ 0 & 0 \end{bmatrix}$$

Bi - bar :

$$= [B^{dil}] + [B^{dev}]$$

$$B^{dil} = \frac{1}{3}\begin{bmatrix} B_1^a & B_2^a \\ B_1^a & B_2^a \\ 0 & 0 \\ B_1^a & B_2^a \end{bmatrix}$$

$$B^a = B^{dil} + B^{dev}$$

$$= \frac{1}{3}\begin{bmatrix} B_1^a & B_2^a \\ B_1^a & B_2^a \\ 0 & 0 \\ B_1^a & B_2^a \end{bmatrix} + \begin{bmatrix} \frac{2}{3}B_1^a & -\frac{1}{3}B_2^a \\ -\frac{1}{3}B_1^a & \frac{2}{3}B_2^a \\ -\frac{1}{3}B_1^a & -\frac{1}{3}B_2^a \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} B_5^a & B_2^a \\ B_4^a & B_3^a \\ B_3^a & B_6^a \\ B_3^a & B_6^a \\ B_7^a & B_{11}^a \\ B_2^a & B_1^a \end{bmatrix}_{4\times 2} \quad a = 1,(1)\,q$$

$$B_4^a = \frac{1}{3}(B_1^a - B_2^a)$$

$$B_6^a = \frac{1}{3}(B_2^a - B_3^a)$$

$$B_7^a = B_3^a + B_6^a$$

$$B_8^a = \frac{B_3^a - B_5^a}{3}$$

$$B_9^a = B_3^a + B_8^a$$

$$B^e = \begin{bmatrix} B^1 & B^2 & B^3 & B^n \\ 4\times2 & 4\times2 & 4\times2 & 4\times2 \end{bmatrix}_{4\times 8}$$

$$[K^e] = \int_{\Omega^e} \underset{8\times4}{B^T} \underset{4\times4}{C} \underset{4\times8}{B} \, d\Omega^e$$

$$C = \begin{bmatrix} c_{11} & c_{12} & 0 & c_{13} \\ c_{12} & c_{22} & 0 & c_{23} \\ 0 & 0 & c_{66} & 0 \\ c_{13} & c_{23} & 0 & c_{33} \end{bmatrix}$$

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THANK YOU !